Smooth Manifolds and Smooth Maps

Chang Yoon Seuk

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1 Motivation

Prior to beginning Milnor's Differential Topology, we need to understand the notion of manifolds. This would be a sufficient factor before starting the further discussions of smooth manifolds and smooth maps. Manifolds in topological spaces consist of three distict properties: Hausdorff Space, second-countability, and locally euclidean of n-dimensional space. Definitions below generally explain criteria of topological manifolds.

Definition 1.1. *Topological Space*
Let X, τ is Topology on X if,

$$1)X, \phi \in \tau,\ 2)A, B \in \tau \Rightarrow A \cap B \in \tau,\ 3)A_i \in \tau \Rightarrow \bigcup_{i \in I} A_i \in \tau$$

Definition 1.2. *Hausdorff Space*

A topological space (X, τ) is Hausdorff, $\forall x, y \in X$ with $x \neq y$
i.e. There is an open neighborhood of $x : U_x$, $y : U_y$ with $U_x \cap U_y = \phi$

Definition 1.3. *Second-Countable Space*
Let (X, τ) be a topological space, then we define a collection of subsets $B \subseteq \tau$ be a basis of τ. If B is countable for its topology, then X satisfies the second countability.

i.e For all open set $U \in \tau$, there is $(A_i)_{i \in I}$ with $A_i \in B$ and $\bigcup_{i \in I} A_i = U$

Definition 1.4. *Locally Euclidean n-dimensional space*
A topological space (X, τ) is defined to be locally Euclidean of dimension n if, $\forall x \in X$, there is an open neighborhood $U \in \tau$ and a homeomorphism $h\colon U \to U'$

Remark (Homeomorphism). Let M, N be topological space, a map $f\colon M \to N$ is homeomorphic, i) f is a bijection, ii)f and f^{-1} are both continous functions.

2 Smooth Manifolds

Definition 2.1. *Smooth Maps*
Let $\mathbb{R}^k$ be k-dimensional Euclidean space, such that $x \in \mathbb{R}^k$ as a tuple.
i.e. $x = (x_1, x_2, , , x_k)$. Then we define $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ be open sets. Here, a map $f\colon U \to V$ is smooth if it is infinitely differentiable.

Remark (Smooth). All the partial derivatives of f exists and its continuous.

Definition 2.2. *Diffeomorphism*
For open sets X and Y, a map f denotes X is diffeomorphic to Y. This requires two conditions: f has a smooth bijection and a smooth inverse.
i.e. $f\colon X \to Y$ is called diffeomorphism, then $\exists g\colon Y \to X$ such that f, g are both infinitely differentiable, and $f \circ g = id_Y$, $g \circ f = id_X$.

Corollary 2.2.1. We could also say that f has a 2-sided inverse. Below the commutative diagram describes the function.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "id_X"'] & Y \arrow[r, "g"] \arrow[rr, bend left, "id_Y"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Definition 2.3. *Smooth manifold*
Let $X \subset R^n$ and $U \subset R^k$ be open sets. For those locally Euclidean spaces R^n and R^k, if for each $x \in X$ has an open neighborhood $W \cap X$, that is diffeomorphic to $U \subset R^k$, then we define a subset X is a smooth manifold of dimension k.

Proposition 2.4. *Parametrization*
A map g in which specifically maps diffeomorphically $U \subset R^k$ to $W \cap X \subset R^n$ is defined as a paramatrization of the region $W \cap X$.
i.e. The particular diffeomorphism $g\colon U \to W \cap X$. Clearly, we could observe the $g(U) = x$,such that $x \in W \cap X$.

Example. Milnor provided S^n, a n-dimensional unit sphere as an example of smooth manifold. Here we define $S^{n-1} = \{x^n | \Sigma {x_i}^2 = 1\} \subset R^n$ a smooth manifold while containing all the tuples $x_i = (x_1, x_2, , , , x_n)$ of locally Euclidean space R^n.
In the case of S^2, the tuples (x_1, x_2) maps diffeomorphically to $(x_1, x_2, x_3) \in R^3$ such as

$$(x_1, x_2) \mapsto (x_1, x_2, \sqrt{1 - x_2^2 - x_3^2})$$

3 Tangent Spaces and Derivatives

Definition 3.1. *Derivative*
Let $U \subset R^k$ such that $x \in U$ and $h \in R^k$,

$$df_x(h) = \lim_{t \to \infty} = \frac{f(x+th) - f(x)}{t}$$

Then a derivative of any smooth map $f : U \to V$ defined as $df_x : R^k \to R^l$.

Definition 3.2. *Tangent Spaces*
Let U, V be manifolds such that $x \in U \subset R^k$; $y \in V \subset R^l$. First, we apply the notion of the derivative above as a linear mapping in any point of $x \in U$, a collection of those directional derivatives is called tangent space: TU_x.
Furthermore, a smooth map $f : U \to V$ such that $x \in U$ and $y \in V$, with $y = f(x)$, we see that $df_x : TU_x \to TV_y$.
Observe from the linear mapping from the given manifold, we could see that those TU_x and TV_y is a submanifold onto R^k and R^l.

Proposition 3.3. *Chain Rule*
Let $U \subset R^k$; $V \subset R^l$; $W \subset R^m$ with $f(x) = y$,such that $x \in U$ and $y \in V$.
Following commutative diagram describes the chain rule.
If $f : U \to V$; $g : V \to W$ are smooth maps, we have

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & W \end{tikzcd}$$

,then apply linear mappings in which derives from the smooth maps above

$$\begin{tikzcd} & R^l \arrow[dr, "dg_y"] & \\ R^k \arrow[ur, "df_x"] \arrow[rr, "d(g \circ f)_x"'] & & R^m \end{tikzcd}$$

Hence, we notice that $d(g \circ f)_x = dg_y \circ df_x$.

Corollary 3.3.1. *Identity mapping*
Let $U \subset R^k$, such that $U \subset U'$ be open sets with $x \in U$.
If, i maps U to U' diffeomorphically, then $di_x = i$.

Proposition 3.4. *Nonsingular mapping*
Let $U \subset R^k$ and $V \subset R^l$, such that $x \in U$ and $y \in V$. If $f\colon U \to V$ is a smooth map with $f(x) = y$, then $k = l$.

Proof. In respect to the commutative representation from the Propositionn 3.3, a chain rule, we have

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[rr, bend left, "id"] & V \arrow[r, "f^{-1}"] & U \end{tikzcd}$$

Then, the composition of a linear mapping shows that

$$\begin{tikzcd} R^k \arrow[r, "df_x"] \arrow[rr, bend left, "id"] & R^l \arrow[r, "df_y^{-1}"] & R^k \end{tikzcd}$$

This follows $k = l$, hence, if $f\colon U \to V$ is diffeomorphism, then df_x is isomorphism. i.e The linear mapping should be invertible as desired. $\square$

Proposition 3.5. *Inverse Function Theorem*
Let $U \subset R^k$ be open sets such that $x \in U$. If $df_x\colon R^k \to R^k$ be a smooth then, $f\colon U' \to U$ is diffeomorphism where $U' \subset U$. i.e. Since we already know that df_x is nonsingular, f maps $x \in U'$ diffeomorphically to an open set $f(U')$.

Remark. A smooth map f need not to be always injective.

4 Regular Values

Definition 4.1. *Regular point*
Suppose two maninfolds $M \subset R^k$ and $N \subset R^k$. If $f.\colon M \to N$ is a smooth mapping then, we call $x \in M$ be a regular point of f. The strict condition applies with an invertible linear mapping df_x.

Definition 4.2. *Regular value*
Since we are mapping through same dimensions that are locally Euclidean, we first set $f(x) = y$,such that $y \in N$. If $f\colon M \to N$ is diffeomorphism, then we could define a regular value matches to each $y \in N$ when $f^{-1}(y)$ corresponds only to regular points $x \in M$.

Remark. This regular values generates from the Proposition 3.5 Inverse Function Theorem.

5 The Fundamental Theorem of Algebra

By definition, we already know the fundamental theorem of Algebra: every nonconstant complex polinomial $P(z)$ must have a zero. However, I sill don't get the Milnor's example of using this notion.

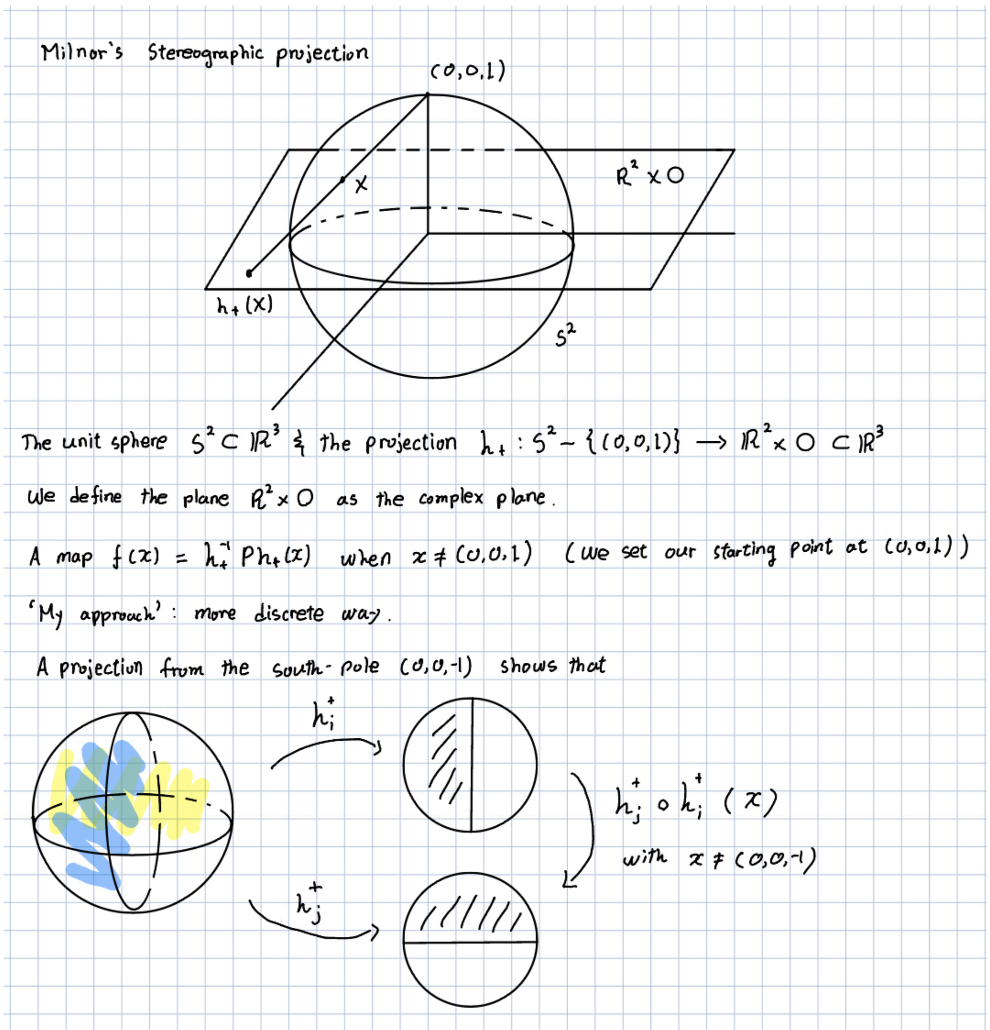


Figure 1: This is a screenshot from my notes.

Supervised Readings I
Differential Topology

Problem Set #1

Instructor: Inbar Klang

Student: Chang Yoon Seuk

Submission: 07/20/2022

Signature: Seuk

Problem 8 Let $M \subset \mathbb{R}^k$ & $N \subset \mathbb{R}^l$ be smooth manifolds.

We want to show $T(M \times N)_{(x,y)} = TM_x \times TN_y$.

Since M & N are both smooth manifolds, $M \times N \subset \mathbb{R}^{k+l}$ is also a smooth manifold.

Now, let $(x,y) \in (M \times N)$, we define tangent space $T(M \times N)_{(x,y)}$ by local parametrization.

$f \times g : U \times V \to M \times N \subset \mathbb{R}^{k+l}$ of a neighborhood $(f \times g)(U \times V)$ of (x,y)

in $(M \times N)$. i.e. There exists $(f \times g)(u,v) = (x,y)$, given by $f(u) = x$, $g(v) = y$.

Here, if we take a map $f \times g$ as a mapping from open subsets U in $\mathbb{R}^m$ and V in $\mathbb{R}^n$

to $\mathbb{R}^{k+l}$, the derivative $d(f \times g)_{(u,v)} : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^{k+l}$.

Then, our tangent space $T(M \times N)_{(x,y)}$ is equal to the Image $d(f \times g)_{(u,v)}(\mathbb{R}^{m+n})$.

Now, consider parametrization of each smooth manifold M and N, such that $x \in M$ and $y \in N$

we have $f : U \to M \subset \mathbb{R}^k$ with $f(u) = x$ & $g : V \to N \subset \mathbb{R}^l$ with $g(v) = y$

Then, we see the derivative $df_u : \mathbb{R}^m \to \mathbb{R}^k$ & $dg_v : \mathbb{R}^n \to \mathbb{R}^l$.

Applying the same notion above, $TM_x =$ Image $df_u(\mathbb{R}^m)$ of df_u ; $TM_y =$ Image $dg_v(\mathbb{R}^n)$ of dg_v.

Notice $TM_x \times TN_y = \text{Image}\,(df_u \times dg_v)(\mathbb{R}^{m+n})$

$= \text{Image}\,(df_u(x), dg_v(y))$ ($\because$ we already know $f(U)$ of x in M & $g(V)$ of y in N)

$= \text{Image}\,d(f \times g)_{(u,v)}(\mathbb{R}^{m+n})$ ($\because$ given $f(u) = x$, $g(v) = y$)

Hence, $T(M \times N)_{(x,y)} = TM_x \times TN_y$ $\square$

Problem 9

i) Graph Γ : Set of all $(x,y) \in M \times N$ with $f(x) = y$, $f: M \to N$ be a smooth map.

We want to show that graph Γ is a smooth manifold.

Let $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$ be open sets, since $f: M \to N$ is a smooth mapping,

for each $x \in M$, $\exists$ open set $W \subset \mathbb{R}^k$ with $x \in W$ & a smooth map $F: W \to \mathbb{R}^l$

in which coincides f throughout $W \cap M$.

Here, consider a parametrization $g: U \to M \subset \mathbb{R}^k$ & $h: V \to N \subset \mathbb{R}^l$,

we observe the commutative map

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbb{R}^l \\ U \arrow[u, "g"] \arrow[r, "h^{-1} \circ f \circ g"'] & V \arrow[u, "h"'] \end{tikzcd}$$

$(\because f = F \text{ in } W \cap M)$

Now, graph Γ is a set of all the sets $(x, f(x))$ in $M \times N$

, and we already know that f maps smoothly $g(U)$ into $h(V)$ for neighborhoods

$g(U)$ of x ; $h(V)$ of $f(x)$.

This means, graph Γ has a subset $X \times Y \subset \mathbb{R}^{k+l}$ such that $x \in X$, $f(x) \in Y$

(i.e. $(x, f(x)) \in X \times Y$) contains a neighborhood $W \cap M$, diffeomorphic to $U \times V$

of the Euclidean space $\mathbb{R}^{m+n}$ (Note : $U \subset \mathbb{R}^m$ & $V \subset \mathbb{R}^n$) .

Hence, graph Γ is a smooth manifold $\square$

ii) $T\Gamma_{(x,y)} \subset TM_x \times TN_y = \text{graph}(df_x)$

Consider commutative gram of the linear mapping, where $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^k$

such that $u = g^{-1}(x)$, $v = h^{-1}(y)$ with $f(x) = y$

we have
$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^l \\ \mathbb{R}^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1}\circ f\circ g)_u"'] & \mathbb{R}^n \arrow[u, "dh_v"'] \end{tikzcd}$$

since $M \subset \mathbb{R}^k$ & $N \subset \mathbb{R}^l$, we see that dF_x maps $\text{Image}(dg_u)$ into $\text{Image}(dh_v)$.

Moreover, df_x does not always depend on the particular F; so,

$df_x = dh_v \circ d(h^{-1}\circ f \circ g)_u \circ dg_u^{-1}$

This gives us the fact : $df_x : TM_x \to TN_y$.

Now, $TM_x \times TN_y = \text{Image}(dg_u \times dh_v)\left(\mathbb{R}^{m+n}\right)$

$= \text{Image}\left(dg_u(x), dh_v(y)\right)$

$= \text{graph}(df_x)$ ...

<u>Problem 10</u> Let $M \subset \mathbb{R}^k$, $TM = \{(x,v) \in M \times \mathbb{R}^k \mid v \in TM_x\}$

i) We want to show TM is a smooth manifold

Since v's are elements of the vector space TM_x ; defined as tangent vectors to M at x, $TM_x \subset \mathbb{R}^k$.

Now, TM is a set of all the sets (x, TM_x) where $x = (x_1, \dots, x_k)$ be a tuple.

For all $x \in M$, df_x carries x homeomorphically to TM_x and both df_x, df_x^{-1} is smooth, it is clearly a diffeomorphism.

Then, we have a subset $T \subset TM$ such that $(x,v) \in T$ has a neighborhood $W \cap T$ in which is diffeomorphic to an openset U in $\mathbb{R}^m$.

Hence, TM is a smooth manifold $\square$

ii) We want to show $f: M \to N$ be a smooth map ; so as to $df: TM \to TN$ where $d(g \circ f) = dg \circ df$

Let $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$ such that $x \in M$ and $y \in N$, if $f: M \to N$ is a smooth, then $k = l$ with $f(x) = y$.

Applying chain with a commutative map,
$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "Id"'] & N \arrow[r, "f^{-1}"] & M \end{tikzcd}$$

Then, a linear map of our composition
$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "df_x"] \arrow[rr, bend right, "Id"'] & \mathbb{R}^l \arrow[r, "df_y^{-1}"] & \mathbb{R}^k \end{tikzcd}$$

So, if $f: M \to N$ is diffeomorphic, df_x is isomorphic.

Now, let $TN = \{(y,u) \in N \times \mathbb{R}^l \mid u \in TN_y\}$.

Since $f(x) = y$, we set $g = f^{-1}_y$, then
$$\begin{tikzcd} TM_x \arrow[r, "df_x"] \arrow[rr, bend right, "Id"'] & TN_y \arrow[r, "dg_y"] & TM_x \end{tikzcd}$$

This implies
$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[rr, bend right, "Id"'] & TN \arrow[r, "dg"] & TM \end{tikzcd}.$$

Hence, a linear map $df: TM \to TN$ is isomorphism $\square$

Supervised Readings I
Differential Topology

Exercises Re-visited

Instructor: Inbar Klang

Student: Chang Yoon Seuk

Submission: 07/27/2022

Signature:

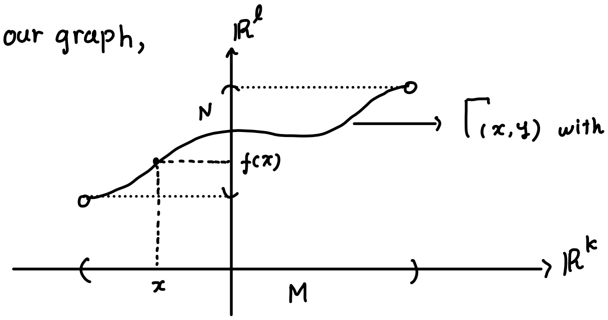
#9

i) We want to show that graph $\Gamma_{(x,y)}$ is a smooth manifold.

Let $f: M \to N$ be a smooth map. Graph $\Gamma_{(x,y)} = \{ (x,y) \in M \times N \mid f(x) = y \}$

Let $M \subset \mathbb{R}^k$ & $N \subset \mathbb{R}^\ell$, locally-Euclidean spaces. Notice our $\Gamma_{(x,y)} \subset \mathbb{R}^{k+\ell}$

Now, visualize our graph,



Then, consider projection map $\pi : \Gamma_{(x,y)} \longrightarrow M$ & $\pi^{-1} : M \longrightarrow \Gamma_{(x,y)}$

$(x, f(x)) \longmapsto x$ $\qquad$ $x \longmapsto (x, f(x))$

Since f is a smooth map, we already know the map contains smooth inverse and bijection.

Indeed, π and π^{-1} are both continuous and bijective along with smoothness, concludes to diffeomorphism. Now, we will use local parametrization to configure $\Gamma_{(x,y)}$ is a smooth manifold.

Let $U, V \subset M \times N$ such that open sets of $\mathbb{R}^{k+\ell}$. i.e. $U \subset \mathbb{R}^{k+\ell}$ & $V \subset \mathbb{R}^m : m \leq k+\ell$.

We set $h \times g : U \times V \longrightarrow M \times N \subset \mathbb{R}^{k+\ell}$ be a parametrization.

i.e. There exists $(h \times g)(U \times V) = (x, y)$ given by $h(U) = x$ and $g(V) = y$.

We have to be careful that $y = f(x)$ where f is diffeomorphism.

Hence, our local parametrization $h \times g$ maps U and V to the neighborhood of x and $f(x)$, it is also diffeomorphism.

Observe the commutative diagram,

$$\begin{tikzcd} M \times N \arrow[r, "\pi_1"] & N \\ U \times V \arrow[u, "h \times g"] \arrow[r, "\pi_2"'] & U \arrow[u, "g"'] \end{tikzcd}$$

The projection map π_1 maps $M \times N$ to our Image $f(M) = N$ smoothly.

Then, $\pi_2 = g^{-1} \circ \pi_1 \circ (h \times g) \ : U \times V \longrightarrow U$ is also a smooth projection.

Hence our projection map π_2 has a smooth bijection and inverse, it is diffeomorphism between open sets $U, V \subset \Gamma_{(x,y)}$.

Thus, we have shown that $\Gamma_{(x,y)}$ is a smooth manifold $\square$

ii) We want to show $T\Gamma_{(x,y)} \subset TM_x \times TN_y$

From $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^l$ open and $f: M \to N$ be smooth map, we will parametrize $\Gamma_{(x,y)}$.

given by $g: M \longrightarrow \Gamma_{(x,y)} \subset \mathbb{R}^{k+l}$. i.e. $g: x \longmapsto (x, f(x))$

Then, our tangent space of the graph $T\Gamma_{(x,y)} = \text{Image}(dg_x)$ such that $\mathbb{R}^k \longrightarrow \mathbb{R}^{k+l}$.

This means $T\Gamma_{(x,y)} = \Gamma df(x) \quad \forall x \in M$.

If we parametrize M and N seperately, such that $h_1: U \to M \subset \mathbb{R}^k$ & $h_2: V \to N \subset \mathbb{R}^l$,

of a neighborhood $h_1(U)$ of $x \in M$ with $h_1(u) = x$ and $h_2(V)$ of $y \in N$ where $y = f(x)$,

with $h_2(v) = y$, then we have the following tangent spaces.

$$TM_x = \text{Image}(dh_1u) \quad \text{and} \quad TN_y = \text{Image}(dh_2v)$$

Now, $TM_x \times TN_y = \text{Image}(dh_1u \times dh_2V)$

$$= \text{Image}(dh_1u(x),\ dh_2v(y)) \ ; \ f(x) = y,$$

$$= \text{Image}(dx \times df(x)) \subset \mathbb{R}^{2k+l}$$

$$(\because \text{Im}(dx): \mathbb{R}^k \longrightarrow \mathbb{R}^k \ \&\ \text{Im}(df(x)): \mathbb{R}^k \longrightarrow \mathbb{R}^{k+l})$$

Hence, $T\Gamma_{(x,y)} \subset TM_x \times TN_y$ $\square$

#10

i) Given $M \subset \mathbb{R}^k$, we define $TM = \{(x,v) \in M \times \mathbb{R}^k \mid v \in TM_x\}$ as a Tangent bundle space.

i.e. $TM = \bigcup_{x \in M} TM_x$. We want to show that TM is a smooth manifold.

Construct a projection map $\pi : TM \longrightarrow M$ that is smooth.
$$(x,v) \longmapsto x$$

i.e. $\forall\, V \in TM_x$, π maps to $\forall\, x \in M$.

Clearly, $\pi^{-1} : M \longrightarrow TM$ is also smooth.
$$x \longmapsto (x,v)$$

Then, we see that our projection map is diffeomorphism.

Now, construct a local parametrization from our projection ; following the commutative gram

$$\begin{tikzcd}
TM \ \pi^{-1}(U) \arrow[rr, "h"] \arrow[d, shift right, "\pi"'] & & \mathbb{R}^{2k} \arrow[d, "h^{-1}\circ(\pi\circ g)"] \\
M \ U \arrow[u, shift right, "\pi^{-1}"'] \arrow[rr, "g"] & & g(U) \ \mathbb{R}^k
\end{tikzcd}$$

Here, if we locally parametrize TM such that $h : \pi^{-1}(U) \longrightarrow TM \subset \mathbb{R}^{2k}$,

it followed by smooth projection π, h is also a smooth map.

Hence, TM is a smooth manifold. $\square$

ii) For any smooth map $f : M \longrightarrow N \Rightarrow df : TM \longrightarrow TN$

i.e. The linear mapping from TM to TN.

First, there must be the existence of 2-sided inverse ; f & g are both infinitely differentiable.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[rr, bend right, "idM"'] & N \arrow[r, "g"] \arrow[rr, bend left, "idN"] & M \arrow[r, "f"] & N
\end{tikzcd}$$

This follows $M \subset \mathbb{R}^k$ & $N \subset \mathbb{R}^k$, such that df is isomorphism.

Supervised Readings I
Differential Topology

Special Exercise

Instructor: Inbar Klang

Student: Chang Yoon Seuk

Submission: 08/05/2022

Signature: Seuk

Let $S^1 \subset \mathbb{R}^2$, consisting of points (x, y), such that $x^2 + y^2 = 1$.

<u>#1</u> We want to show that S^1 is a smooth manifold.

Since $S^1 \subset \mathbb{R}^2$, a locally-Euclidean space, at any point $p \in S^1$ is a 2-tuple $p = (x, y)$. Now, from a local parametrization of S^1, such that $U \subset S^1$ with $p \in U$.

$f : U \longrightarrow M \subset \mathbb{R}^k$ $(k \leq 2)$, for each partial derivatives $(\partial f/\partial x_i, \partial f/\partial y_i)_{i \in I}$ exists.

Explicitly, we could consider 3 cases from $P = (x, y) \in S_1$

i) If $x = 0$, then we have a map $g : Y \longrightarrow X$ for $x \in X$ & $y \in Y$

with $g_1(y) = \sqrt{1 - x^2}$ or $g_2(y) = -\sqrt{1 - x^2}$.

Notice g carries Y to X homeomorphically, and the graph $\Gamma g = S^1$

ii) If $x > 0$, then $h : Y \longrightarrow X$ with $h(y) = \sqrt{1 - y^2}$

$y \longmapsto \sqrt{1 - y^2}$

From the open subset $y \in (-1, 1) \subset \mathbb{R}$, h has smooth bijection and inverse.

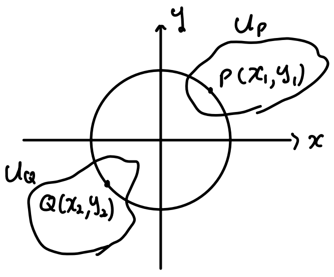
The graph $\Gamma h = S^1$

iii) If $x < 0$, then we have a smooth map $H : Y \longrightarrow X$ with $H(y) = -\sqrt{1 - y^2}$

Using the result from ii), for any $y \in (-1, 1)$, H and H^{-1} is smooth and bijective.

Moreover, because $S^1 \subset \mathbb{R}^2$ and we could construct 2nd-countable basis from $\mathbb{R}^2$, S^1 is also 2nd-countable.

Also, we can construct a openset in any points of S^1, such that $P(x_1, y_1), Q(x_2, y_2) \in S^1$.

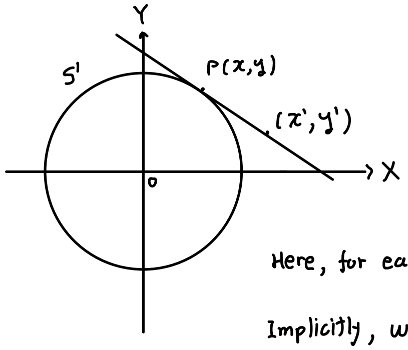


For any $P \neq Q$, there exists open neighborhoods $P : U_{x_1, y_1}$, $Q : U_{x_2, y_2}$.

Then, we find $U_P \cap U_Q = \phi$, and thus it is Hausdorff.

To conclude, S^1 satisfies the properties of manifolds along with smoothness, it is a smooth manifold $\square$

#2 Tangent space $TS'_{x,y}$. Let $p \in S'$ such that $p = (x,y)$.



From a local parametrization of $S' \subset \mathbb{R}^2$

We have shown that $g: x \in X \subset \mathbb{R} \longrightarrow S' \subset \mathbb{R}^2$

$x \longmapsto (x, y)$

Clearly S' is a smooth manifold of $\mathbb{R}$.

Here, for each $p \in S' \subset \mathbb{R}^2$, the derivative is $dg_p : \mathbb{R} \longrightarrow \mathbb{R}$

Implicitly, we have a tangent line at (x,y) : $dx + dy = 0$.

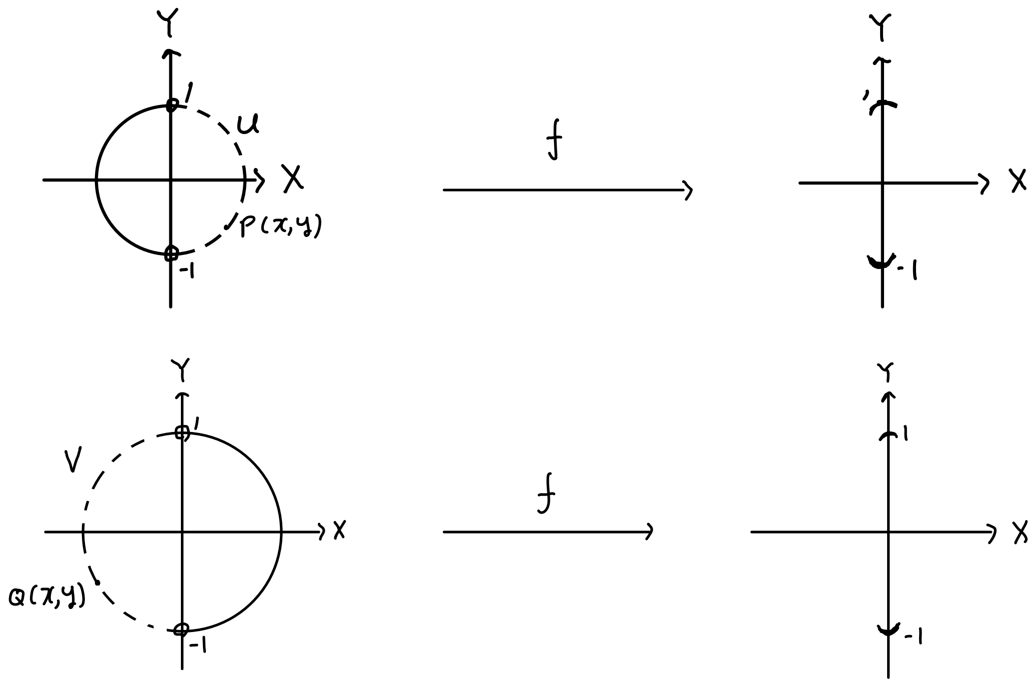
Precisely at a point (x,y), $TS'_{x,y} \subset \mathbb{R}^2$ of dimension 1 is the tangent space of S'.

#3 We want to show $f: S' \longrightarrow \mathbb{R}$ such that $f(x,y) = y$ is a smooth map.

Initially, we will construct 2 open sets $U, V \subset S'$ such that $U \cap V = \phi$.

Let $U = \{ \forall P(x,y) \in S' \mid x > 0 \}$ & $V = \{ \forall Q(x,y) \in S' \mid x < 0 \}$

Then our map $f: S' \to y$ describes by the following :



i) Case U : we have $f(U) = y$, $f^{-1}(y) = \left(\sqrt{1-y^2}, y\right)$

The derivative $df_u = dy$ & $df^{-1}_y = \left(\frac{-y}{\sqrt{1-y^2}}, 1\right)$

ii) case V : We have $f(V) = y$, $f^{-1}(y) = \left(-\sqrt{1-y^2}, y\right)$

The derivative $df_v = 1$ & $df^{-1}_y = \left(\frac{y}{\sqrt{1-y^2}}, 1\right)$

Notice, f and f^{-1} are continuous and bijective ; hence it is homeomorphism.

Also, each coordinate $P \in U$ & $Q \in V$ has derivatives .

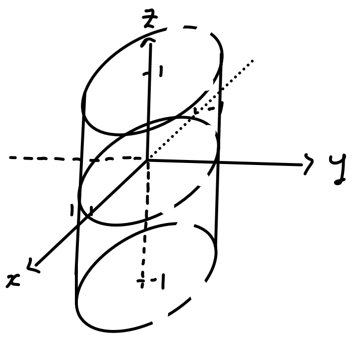
Thus, f is a diffeomorphism ; so, it is a smooth map.

#4 From a smooth map $f: S^1 \longrightarrow \mathbb{R}$ with $f(x,y) = y$, the grap Γf is defined by $\Gamma_{(x,y)} = \{ (x,y) \in S^1 \times \mathbb{R} \mid f(x,y) = y \}$.

We already know that S^1 is a smooth manifold of 1-dimension.

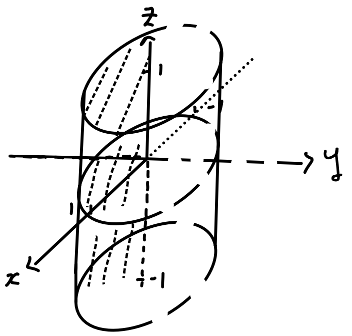
i) From case U : $\Gamma f_u = \{ (x,y), \sqrt{1-y^2} \}$

Right part of $S^1 : x^2+y^2 = 1$ with height $(-1,1)$



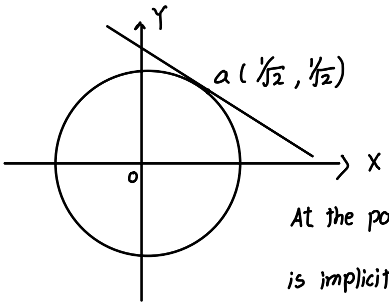
ii) Likewise, from V : $\Gamma f_v = \{ (x,y), -\sqrt{1-y^2} \}$

Left half of S^1 : $x^2+y^2 = 1$ with $h \in (-1,1)$.



Hence, our graph $\Gamma(f)$ is 2-dimension manifold in $\mathbb{R}^3$.

#5



since $f(x,y) \to y$

we find $f(a) \to 1/\sqrt{2}$

At the point $(1/\sqrt{2}, 1/\sqrt{2})$, the tangent line to S^1

is implicitly given by $\frac{1}{\sqrt{2}}dx + \frac{1}{\sqrt{2}}dy = 0$

In coordinates $(x', y') = \left(\frac{1}{\sqrt{2}} + dx, \frac{1}{\sqrt{2}} + dy\right) \Rightarrow x' + y' = \sqrt{2}$

So, the derivative $df_a : \mathbb{R} \to \mathbb{R}$ maps each $x \in X$ onto $\frac{1}{\sqrt{2}} + dx$ & $y \in Y$ onto $\frac{1}{\sqrt{2}} + dy$

Observe that the graph $\Gamma df_a \subset \mathbb{R}^2$ is the line $x + y = \sqrt{2}$.

Also, Γf_a is the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ such that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \longrightarrow \frac{1}{\sqrt{2}}$.

Then, the tangent space $T\Gamma f_a$ is a line that passes $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \perp y = x$.

Hence, $\Gamma df_a = T(\Gamma f_a)$ $\square$

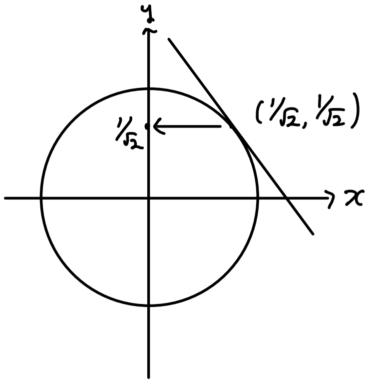
#e

$a = (1/\sqrt{2}, 1/\sqrt{2}) \in S^1$

$f : S^1 \longrightarrow \mathbb{R}$

such that $f(x,y) = y$

since $f : S^1 \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$



Notice, $TS_1 \subset \mathbb{R}^2$ of dimension 1.

So, df_a is a linear map from TS^1_a to $T\mathbb{R}_y$ where $y = f(a)$

Then, $df_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "df_y^{-1}"] & \\ \mathbb{R}^2 \arrow[ur, "df_a"] \arrow[rr, "\text{identity}"'] & & \mathbb{R}^2 \end{tikzcd}$$

The derivative for a smooth map

we find: $df(x,y) : 2x\,dx + 2y\,dy = dy$

Then, at the point $(1/\sqrt{2}, 1/\sqrt{2})$, $f(1/\sqrt{2}, 1/\sqrt{2}) \longrightarrow 1/\sqrt{2}$

; the derivative is $2 \cdot (1/\sqrt{2})\,dx + 2 \cdot (1/\sqrt{2})\,dy = dy$

or explicitly, $\sqrt{2}(x - x') + \sqrt{2}(y - y') = y - y'$

Sard and Brown

Chang Yoon Seuk

August 05, 2022

1 Introduction

Prior to studying Sard's theorem, we will first review the terms of rank in smooth maps, regular values, and density. As of preview, the theorem explains that the smooth map from the set of critical values to other Euclidean space is a Lebesque measure zero. Throughout history, the basis of the preliminary theorem was discovered by Arthur Brown in 1935, extended by Morse in 1939, then Sard proved the theorem presented in the American Mathematical Society in 1942. In turn, it is also known as Sard-Morse-Brown theorem of analyzing the smooth map of critical set. Our goal here is to show that why does Brown theorem is a corollary followed by Sard.

Definition 1.1. *Rank in a smooth map*
In general, let $M \subset R^k$ and $N \subset R^l$, such that $x \in M$; $y \in N$ with $f(x) = y$, if $f : M \to N$ be a smooth map between smooth manifolds M and N, then we see the derivative $df_x : TM_x \to TM_y$. Here, rank of a smooth map is equal to the rank of its differential at a point $x \in M$.
i.e df_x has $rank(k)$ so that the $image(TM_x)$ has dimension k.

Definition 1.2. *Regular values*
Let $M \subset R^m$ and $N \subset R^n$, we have a smooth map $f : M \to N$. If for any $x \in M$, such that the derivative df_x is invertible, we call it as a regular point of smooth map. Followed by the inverse function theorem, where f may not be always injective; $\exists y \in N$ with $f(x) = y$ if our inverse map $f^{-1}(y)$ has only regular points, then we define the point $y \in N$ to be regular value.
i.e. From a smooth map $f : M \to N$ with $x \in M$; $y \in N$, such that $\forall x \in f^{-1}(y)$, the linear map $df_x : TM_x \to TN_y$ is surjective.
Further observation: Let $\#f^{-1}(y)$ be a collection of regular points, $f^{-1}(y) \in M$ with $y \in N$ be a regular value. Here, if M is compact manifold from a smooth map $f : M \to N$, then the set of regular points $f^{-1}(y)$ is finite. Since, both M and N are manifolds in a smooth map f, there must be a Hausdorffness, second-

contablility, and locally-Euclidean as a property of topological manifolds. Then we could consntruct some pairwise disjoint neighborhodds $U_1, \cdots, U_k$ such that $\bigcup_{k \in I} U_k \subset M$ from a set of regular points $\#f^{-1}(y)$. Likewise, $\exists V_k \subset N$ disjoint neighborhoods such that for any $y' \in V$, we have $\#f^{-1}(y') = \#f^{-1}(y)$. Observe that each neighborhood a map $U_k \in M$ to $V_k \in N$ is diffeomorphism, then we can take $V \in N = \bigcap_{k \in I} V_k - f(M - \bigcup_{k \in I} U_k)$.

Remark. Compactness
Let (X, τ) be a topological space, such that a collection of open subset $A_k \subset X$ be open covering X. i.e. $\bigcup_{k \in I} A_k = X$.
Then, X is compact whenever $\forall A_k$ contains finite subcollection that also covers X. i.e. $\forall a_j \subset A_k$, then $\bigcup_{j \in I} a_j = X$

Definition 1.3. *Critical values*
From our smooth map $f : M \to N$, if $\exists x \in M$ such that df_x is singular, we call it as a critical point of f and the image $f(x) \in N$ is a critical value.
i.e. The derivative from a smooth map f at a point $x \in M$, $df_x : TM_x \to 0$.

Definition 1.4. *Density*
Recall from a Basic Topology, let (X, τ) be a topology on X; let $U \subset X$ be a subset. If $\overline{U} = X$, then it is everwhere dense. Also, U is dense in X if and only if non-empty open subset $V \subset X$, such that $U \cup V \neq \phi$. However, we need to expand this notion toward manifold.

2 Sard's Theorem

Theorem 2.1. *Sard, 1942*
Let $U \in R^m$ be ann open set, let $f : U \to R^n$ be a smooth map.
If

$$C = \{x \in U | \, rank \, df_x < n\}$$

, then the image $f(C) \subset R^n$ has Lebesgue measure zero.

Remark. Lebesgue measure zero
Milnor briefly explained that for any $\varepsilon > 0$, to cover image$f(C)$ by a sequence of cubes in R^n with total n-dimension volume is less than ε.
i.e. For $f(C) \subset R^n$, if the image $f(C)$ is Lebesgue measurable zero, we define as

$$m(f(c)) = \sum Vol_n B_i < \varepsilon$$

(the sum of open cubes B_i in total of n-dimension).

Corollary 2.1.1. *Brown, 1935*
Let $M \subset R^m$, $N \subset R^n$, from a smooth map $f : M \to N$, the set of regular values is everywhere dense in N.

Now, we want to show the analysis of Sard's theorem contains the fact of Brown's. First, we will use some concrete example in order to demonstrate Sard's theorem.

Supervised Readings I
Differential Topology

Special Exercise Re-visited

Instructor: Inbar Klang

Student: Chang Yoon Seuk

Submission: 08/19/2022

Signature:

#b

At the point (x,y) in the smooth manifold S^1

First, we find the derivative : $dS^1_{x,y} = 2x\,dx + 2y\,dy = 0$; the slope $= -\frac{x}{y}$

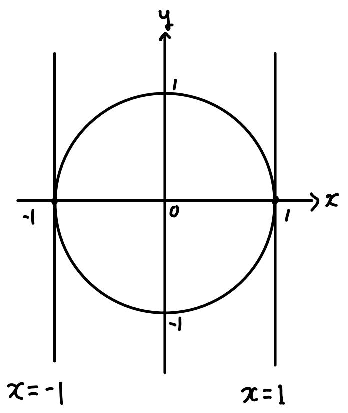
Let (x,y) be $(x',y') \in S^1$, the equation of the linear line

$$y - y' = -\frac{x}{y}(x - x')$$

Explicitly, we have $2y(y-y') + 2x(x-x') = 0$

This is $TS^1_{x,y} \subset \mathbb{R}^2$ of dimension 1.

i) $y = 0$ with $x < 0$ or $x > 0$, then $TS^1_{x,y}$ is the following :

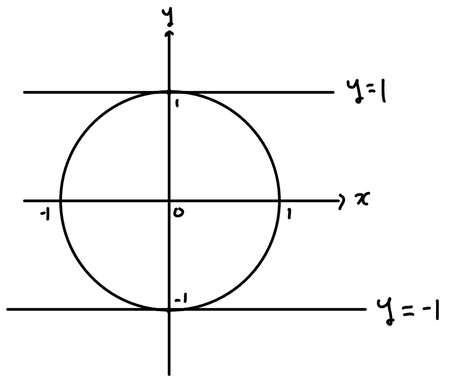


For $x > 0$, $TS^1_{x,y}$ is the line $x = 1$

Likewise, for $x < 0$, $TS^1_{x,y}$: $x = -1$

The slope : $-\frac{x}{y} = \infty$

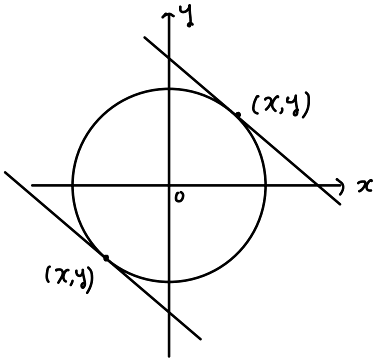
ii) $x = 0$, then $(x,y) = (0,1)$ or $(0,-1)$



since $\frac{dy}{dx} = -\frac{x}{y} = 0$,

$TS^1_{x,y}$ is the linear line $y = \pm 1$

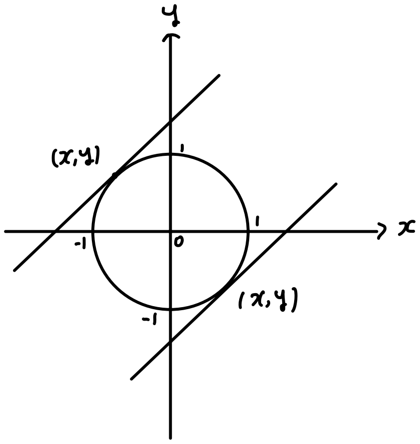
iii) $x>0, y>0$ or $x<0, y<0$



Our $dy/dx = -\frac{x}{y} < 0$; so,

the following linear line shows $TS'_{x,y}$

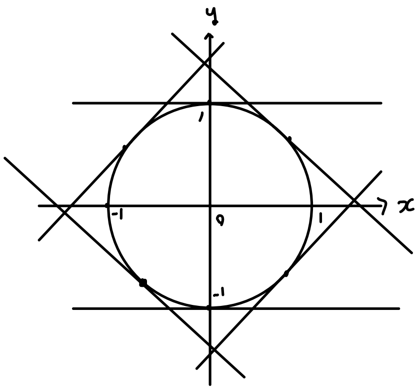
iv) $x<0, y>0$ or $x>0, y<0$



$TS'_{x,y} \subset \mathbb{R}^2$ with $\frac{dy}{dx} = -\frac{x}{y} > 0$

from this case.

Combine all the cases, $TS'_{x,y}$ is the linear line for all $(x,y) \in S'$



#C

We want to show $f: S^1 \longrightarrow \mathbb{R}$ with $f(x,y) = y$ is a smooth map.

Let $U \subset S^1$, $V \subset \mathbb{R}$ then we have the following commutative diagram

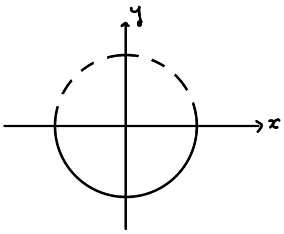
$$\begin{tikzcd} & V \arrow[dr, "f^{-1}"] & \\ U \arrow[ur, "f"] \arrow[rr, "f^{-1} \circ f"'] & & U \end{tikzcd}$$

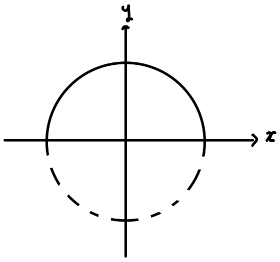
This gives rise to the derivative, since $U \subset \mathbb{R}^2$ & $V \subset \mathbb{R}$

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "df^{-1}_y"] & \\ \mathbb{R}^2 \arrow[ur, "df_{(x,y)}"] \arrow[rr, "\text{Identity}"'] & & \mathbb{R}^2 \end{tikzcd}$$

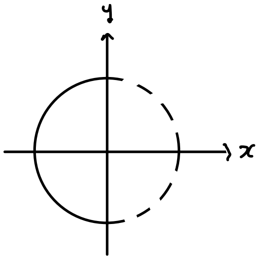
Explicitly, construct open sets in S^1 with the following :

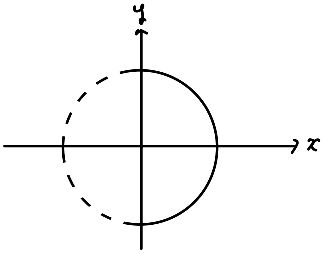
$U_1 = \{ (x,y) \in S^1 \mid y > 0 \}$, $U_2 = \{ (x,y) \in S^1 \mid y < 0 \}$





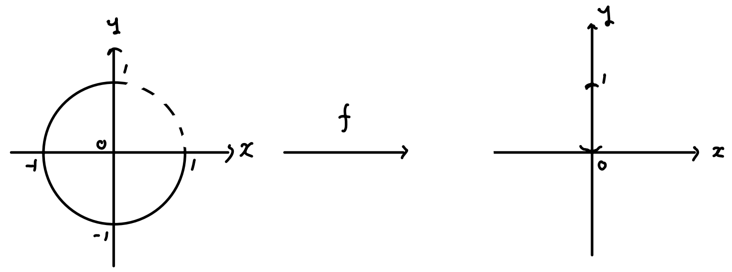
$U_3 = \{ (x,y) \in S^1 \mid x > 0 \}$, $U_4 = \{ (x,y) \in S^1 \mid x < 0 \}$





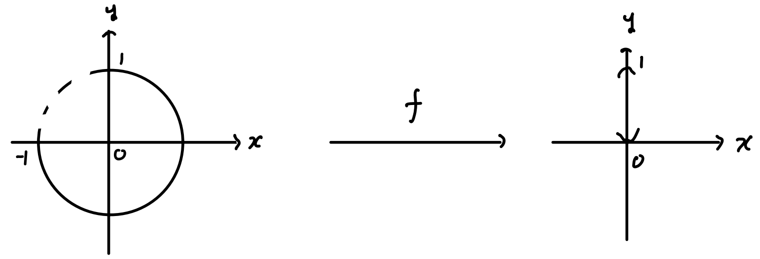
Observe the intersections $U_1 \cap U_3$, $U_1 \cap U_4$ & $U_2 \cap U_3$, $U_2 \cap U_4$

Let $W_1 = U_1 \cap U_3$, then $f(x,y) \longrightarrow y$ shows



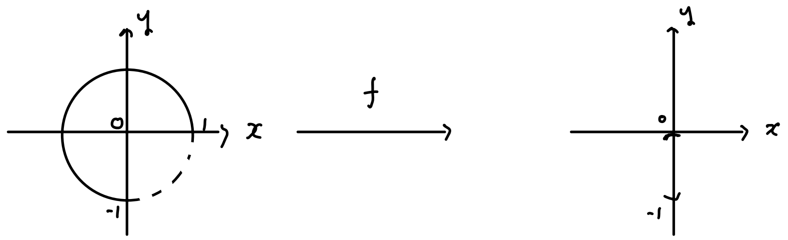
we observe that $f(W_1) \longrightarrow (0,1) \subset \mathbb{R}$

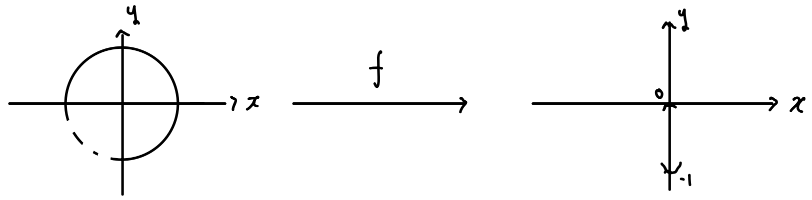
Likewise, $W_2 = U_1 \cap U_4$ then $f(W_2) \longrightarrow (0,1)$



Setting, $W_3 = U_2 \cap U_3$ & $W_4 = U_2 \cap U_4$

$f(W_3) \longrightarrow (0,-1) \subset \mathbb{R}$ & $f(W_4) \longrightarrow (0,-1)$



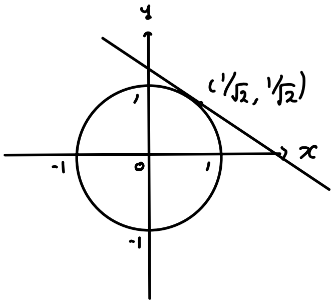


Each W_1, W_2, W_3, W_4 shows homeomorphism and partial derivatives exist.

Hence $f: S^1 \longrightarrow \mathbb{R}$ is a smooth map $\square$

#e The point $a = (1/\sqrt{2}, 1/\sqrt{2}) \in S^1$

First, we will find the tangent space TS^1_a ; described as



$$y - y' = -\frac{x'}{y'}(x - x')$$

Here, $(x', y') = (1/\sqrt{2}, 1/\sqrt{2})$

Then, our TS^1_a : $x + y = \sqrt{2}$

From a smooth map $f : S^1 \longrightarrow \mathbb{R}$, $df_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Since $f(x,y) = y$, we find $df_a : TS^1_a \longrightarrow \mathbb{R}$ such that $df(x,y) = dy$

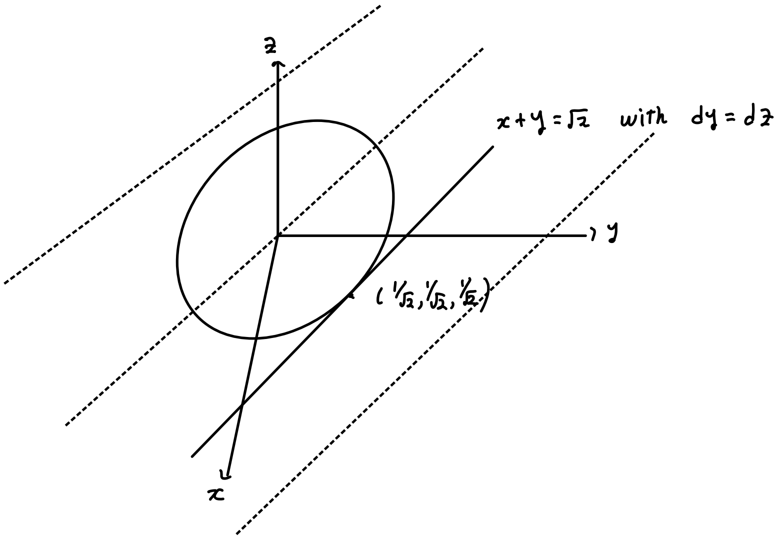
i.e. the tangent space $TS^1_{(1/\sqrt{2}, 1/\sqrt{2})}$ maps to $\mathbb{R}$ in y-axis .

Then, the graph $\Gamma(df_a) \subset \mathbb{R}^3$ is $(x + y = \sqrt{2}, dy = dz)$.

So, the linear line $x + y = \sqrt{2}$ on a plane $y = z$.

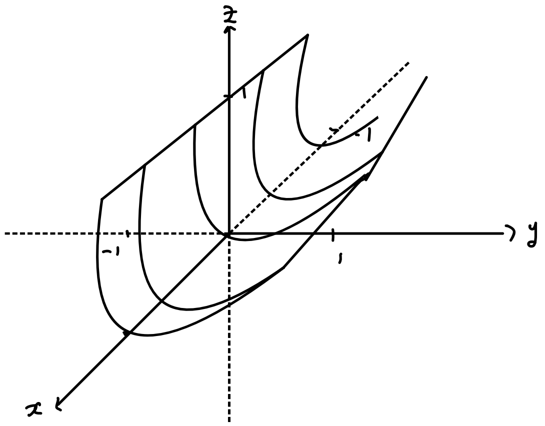
Since the $\Gamma_{a, f(a)}$ is $S^1 = x^2 + y^2 = 1$ on a plane $y = z$,

$T\Gamma_{a, f(a)}$ is the linear line that passes $(1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$; lying on the plane $y = z$



#C-2 Consider $g:(x,y)\longrightarrow \mathbb{R}$ defined by $g(x,y)=y^2$

Since $(x,y)\in S^1$, we find the graph $\Gamma g(x,y)=(x,y,y^2)$ as the following



At the point $a(1/\sqrt{2}, 1/\sqrt{2})$: $g(a)=1/2 \Rightarrow \Gamma g(a)=(1/\sqrt{2}, 1/\sqrt{2}, 1/2)$

$dg_a : TS^1_a \longrightarrow \mathbb{R}$ such that $dy=2y$

This means $TS^1_a : x+y=\sqrt{2} \longrightarrow 2y$ with $g(a)\in 2y$

$\Gamma dg_a = (x+y=\sqrt{2}, 2y)$ is the linear line $x+y=\sqrt{2}$ on a plane $z=2y$