THE WEYL INTEGRATION FORMULA

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Abstract. We demonstrate the Weyl integration formual on class functions. In fact, this is the fundamental integration formula on Lie theory that reduces the integrals of conjugacy funtions which are invariant measure on the compact Lie group over the maximal torus. Ultimately, this formula stretches to the Weyl Character formula. We will built from the essential properties and then derive through the integration formula.

Contents

1. Introduction

The *Weyl integration formula* is the integration formula of the class functions on a compact Lie group. In this paper, we want to show the following theorem:

Theorem 1.1 (main)**.** *From G be a compact Lie group, f be any class function on G, and $d\mu, d\nu$ be the normalized Haar measrue on G, T, respectively, then we have the integration formula*

$$\int_G f(\mu)d\mu = \frac{1}{|W|}\int_T f(\nu)|det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right)d\nu$$

Such further demonstration is heavily from chapter 6 of Lectures on Lie Groups by Adams [1] and lecture series of Wang [3]. The author also inspired from chapter 1 and 17 from Lie Groups by Bump [2], but it is too advanced for the author's current progress, so the detailed exposition has been omitted. In order to explicitly observe the formula, we shall provide the initial definitions and basic set up as follows. For a notational consistency, let $\mathfrak{g}, \mathfrak{t}$ be the $Lie(G), Lie(T)$, respectively(Lie algera of the Lie group G and the maximal torus $T \subset G$).

Definition 1.2. The quotient group $N(T)/T$ is called the Weyl group of G. $N(T)$ here is the normalizer of the maximal torus T of the Lie group G, which is

$$N(T) = \{\mu T\mu^{-1} = T : \text{ for any } \mu \in G\}$$

Definition 1.3. The *orbit* of the Lie group G through the manifold $m \in M$ is

$$G \cdot m = \{\mu \cdot m | \mu \in G\} \subset M$$

, and the stabilizer(i.e.the isotropic group) of $m \in M$ is then

$$G_m = \{\mu \cdot m = m | \mu \in G\}$$

Then, the smooth action from G to M is transitive if we have only a single orbit.i.e. $G \cdot m = M$ for any $m \in M$. We call this a *homogenous G-manifold.*

From a basic Lie theory, we have seen that the $G-invariant$ inner product on $\mathfrak{g}$ is a bilinear form $\langle\cdot,\cdot\rangle$, satisfying

$$\langle Ad(\mu)x, Ad(\mu)y\rangle = \langle x, y\rangle \text{ for all } \mu \in G \text{ , } x, y \in \mathfrak{g}$$

where $Ad(\mu)$ is just an adjoint representation $Ad(\mu)x = \mu x \mu^{-1}$. Then we show the basic set up. The quotient group G/T is a *homogenous G-manifold* with tangent space defined as $T_{eT}(G/T) = \mathfrak{g}/\mathfrak{t} = \mathfrak{p}$. In this sense, we may fix an adjoint inner product on $\mathfrak{g}$ and $\mathfrak{p}$ to be the orthogonal complement of $\mathfrak{t} \subset \mathfrak{g}$. Which means, we have the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$;so, $\mathfrak{p} \subset \mathfrak{g}$. Such an adjoint invariance ensures this decomposition to be well-defined. Now we show the key-steps of the proof of the *Weyl integration formula.*

(1) Parametrize the function, denoted Φ to Ψ
(2) Jacobian computation for the change of variables
(3) Observe the regular points
(4) Use the lemma on a density
(5) Obtain the integration formula on class functions

In the process, we begin with the *Haar measure* in the following section.

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2. The Haar Measure

To understand the *Weyl integration formula* we need to first understand the *Haar measure.* Briefly speaking, from the Haar integrable funtion f on Lie group G, we know how to integrate a function on a manifold. We start with a fixed volume and this requires the manifold to be orientable. Clearly, any Lie group is orientable since the tangent bundle $TG \simeq G \times \mathfrak{g}$ is trivial; so, the volume always exists in G. We begin by showing the definition of *left invariance.*

Definition 2.1. A volume form ω on Lie group G is called *left invariant* if $L_g^*\omega = \omega$ for all $g \in G$. i.e. No matter which position we choose for a Lie group, the way of measuring the volume does not change.

Definition 2.2. The *Haar Measure* is an invariant measure defined on $Borel - \sigma$ algebra

Remark 2.3. $Borel - \sigma$ algebra is a $\sigma - algebra$ generated by Borel sets. Here $Borel - \sigma$ sets is the smallest collection of subsets of topological space. Let (X, τ) be topological spaces, where $\tau \subseteq \mathcal{P}(X)$ be the collection of open sets. Then, for any $E \in \mathcal{B}(X)$,

$$\text{Borel set, denoted } \mathcal{B}(x) = \{\text{opensets of } X;\ \bigcup_{i=1} E_i;\ \bigcap_{i=1} E_i;\ \text{complements:} X \setminus E \in \mathcal{B}(X)\}$$

Now, we show the uniqueness and existence of the *Haar measure.*

Theorem 2.4. *Followed by the definition(2.1), for any Lie group G, a left invariant volume form exists, and it is unique up to multiplicative constant.*

Proof. From a Lie group G, we take nonzero basis of n-form, such that, $\omega_e \in \Lambda^n T_e^* G$ be an nonzero element, where $dim(G) = n$. For any $\mu \in G$, let $\omega_\mu = L_{\mu^{-1}}^* \omega_e$. i.e.By left translation, we move the volume form from the identity e to any point $\mu \in G$. Then for all $\mu, \nu \in G$, we compute

$$(L_\mu^*\omega)_\nu = L_\mu^*\omega_{\mu\nu} = L_\mu^*(L_{(\mu\nu)^{-1}}^*\omega_e) = L_\mu^* L_{\nu^{-1}\mu^{-1}}^*\omega_e$$

and since

$$L_\mu^* L_{\nu^{-1}\mu^{-1}}^*\omega_e = \left(L_{(\nu^{-1}\mu^{-1})} \circ L_\mu\right)^* \omega_e = L_{\nu^{-1}}^*\omega_e = \omega_\nu$$

Hence, we see the left invariance

$$\left(L_\mu^*\omega\right)_\nu = \omega_\nu \text{ , which implies } L_\mu^*\omega = \omega$$

Thus, we have shown the existence of the volume form. Next, we want to show the uniqueness. Suppose now that ω' be another left invariant form on Lie group G. We already know that $\Lambda^n T_e G$ is 1-dimensional, and there exists nonzero constant, such that, $C \in \mathbb{R}^\times$, so that $\omega'_e = C_{\omega_e}$. Clearly, from the result above on existence, for any $\mu \in G$, we have the left invariance

$$\omega'_e = L_{\mu^{-1}}^*\omega'_e = C L_{\mu^{-1}}^*\omega_e = C\omega_e$$

Hence, we have shown the $\omega' = C\omega$ everywhere on G, so that the left invariant volume form in unique up to multiplicative constant. $\square$

Then by the definition(2.2) and the result of from the theorem above, we can explicitly show what it means to be the *Haar measure*. We will use differential form and integration, in order to define the measure, denoted m on the Lie group G.

Definition 2.5. Assume that ω is positive and compatible with the orientation of Lie group G. Let us denote C_cG be the space of compactly supported conitunous functions on G.i.e.

$$C_cG = \{f : G \to \mathbb{R}|\ f \text{ is continous and } supp(f) \text{ is compact}\}$$

where,

$$supp(f) = \{\overline{\text{For any } \mu \in G | f(\mu) \neq 0}\}$$

For any conitunous function $f \in C_cG$, we define a measure on G via

$$f \in C_c(G) \mapsto \varphi(f) = \int_G f(\mu)\omega(\mu)$$

Observe that this measure is left invariant from the fact that for any fixed $\nu \in G$, we see that

$$\varphi(f) = \int_G f\omega = \int_G L_\nu^*(f\omega) = \int_G (L_\nu^* f)\omega = \varphi(L_\nu^* f)$$

So then, we call such left invariant measure to be a left *Haar measure*. In particular, by the definition(2.2), let $E \subset G$ be any Borel set. Then we see the left invariant measure m for any $\nu \in G$, such that,

$$m(E) = m(L_\nu E)$$

Definition 2.6 (Normalization)**.** Suppose that we have a compact Lie group G. By the left invariance on differential form, for any $\mu, \nu \in G$, we know that

$$d(\mu\nu) = d\mu \text{ ,or equivalently, } \int_G f(\nu\mu)d\mu = \int_G f(\mu)d\mu \text{ for any fixed } \nu \in G$$

Then, if we denote $\omega = d\mu$ for any $\mu \in G$, then the *Haar measure* ω is normalized if

$$Vol(G) = \int_G \omega = 1$$

Also, on any compact Lie group G, there exists a unique normalized left *Haar measure*.

3. The Weyl Integration Formula

Understanding the *Haar measure*, we are now ready to demonstrate the main theorem. To begin with, we will first show the lemma.

Lemma 3.1. *There exists a normalized density $d(\mu T) = d\mu/d\nu$ on the quotient G/T, which is invariant under $G-$action.*

Remark 3.2. Since the Lie group defines on a smooth structure, we may express the *Haar measure* in local coordinates as a (*density*) $\times$ (standard Lebesque measure). i.e. The *Haar measure*: $d\mu = f(\mu) \cdot dx$, where $f(\mu)$ is a density and dx be the Lebesque measure.

Theorem 3.3. *Let G be a compact Lie group and f be its class functions. Let $d\mu, d\nu$ be the normalized Haar measure on G and T, respectively. Then, we have the formula on class functions,*

$$\int_G f(\mu)d\mu = \frac{1}{|W|}\int_T f(\nu)|det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right)|d\nu$$

The strategy of the proof is as follows. It consists of two parts: parametrization of the function near base points and Jacobian computations, and the observation of the regular points followed by the lemma above. The former one observes the conjugacy classes and shows how the group twists around the maximal torus; the latter version inspects the regular points for a geometric interpretation. Combining all, we will get to the resolution of the proof.

Proof. Consider the map that is not currently volume preserving

$$\Phi : G/T \times T \to G$$

, where

$$(\mu T, \nu) \mapsto (\mu\nu\mu^{-1}) \text{ for any } \mu \in G, \nu \in T$$

For a simpler case, we take the computation near the identity: (eT, e). Then fix μ, ν and construct a parametrization, such that,

$$\Psi : G/T \times T \to G \text{ defined by } (hT, \xi) \mapsto (h\nu\xi h^{-1}\xi^{-1})$$

We observe Ψ in the composition map via

$$\Psi = R_{\nu^{-1}} \circ C(\mu^{-1}) \circ \Phi \circ (\widetilde{L_\mu} \times L_\nu)$$

, and compute each part explicitly. Start with $R_{\nu^{-1}}$, and this is clearly a right translation(i.e.multiplication) by ν^{-1}. It sends our element x near ν to the identity by $x \mapsto x\nu^{-1}$. Next, $C(\mu^{-1})$ is the Jacobian element coming from the left multiplication of μ^{-1}. i..e.We can view this as the conjugacy by μ^{-1} and differential at the point $\mu\nu\mu^{-1}$. In fact, this is a volume preserving since the *Haar measure* under conjugation is invariant. Φ here is a natural conjugation map, which sends the point, such that, $(\mu hT, \nu\xi) \mapsto (\mu h)(\nu\xi)(\mu h)^{-1}$. Finally, the last part consists of two parts: $\widetilde{L_\mu}$ and L_ν. The former part, $\widetilde{L_\mu}$ is a left multiplication by μ on homogenous G/T and the latter is the left multiplication by ν on maximal torus T, which we already know. Recall by the lemma above that the density on G/T given by $d\mu/d\nu$ is invariant under $G - action$; it is clealry a measure preserving in turn. So then, the map $\widetilde{L_\mu} \times L_\nu$ sends a point $(hT, \xi) \mapsto (\mu hT, \xi)$. Notice that the Jacobian here shows how the volume at G splits into the quotient spaces $\mathcal{G}/T$ and T. i.e. The $|det\Phi|$ transforms the *Haar measure* on G to G/T and T. Also, as we already know $d\mu$ and $d(\mu T)$ are $G - invariant$, its corresponding Jacobian must be 1. In short, each of the map preserves the volume by the invariance of the *Haar measure*. Henceforth, we take the differential near the base point (eT, e)

$$d\Psi_{(eT,e)} = dR_{\nu^{-1}(\nu)} \circ dC(\mu^{-1})_{(\mu\nu\mu^{-1})} \circ d\Phi_{(\mu T,\nu)} \circ d(\widetilde{L_\mu} \times L_\nu)_{(eT,e)}$$

Each differential part has determinant of 1, which means,

$$|det(dR_{\nu^{-1}(\nu)})| = |det\left(dC(\mu^{-1})_{(\mu\nu\mu^{-1})}\right)| = |det\left(d(\widetilde{L_\mu} \times L_\nu)_{(eT,e)}\right)| = 1$$

We also nothice that

$$|det\left(d\Phi_{(\mu T,\nu)}\right)| = |det\left(d\Psi_{(eT,e)}\right)| = 1$$

Then, we compute the differential of the parametrization map Ψ at (eT, e), given by the conjugation for any $\mu \in G$ with lineariation of the path. Let $X \in (\mathfrak{g}/\mathfrak{t})$ and $S \in \mathfrak{t}$, where $\mathfrak{g} = Lie(G)$ and $\mathfrak{t} = Lie(T)$, respectively. We consider the tangent vector near the origin (eT, e) as $(X, S) \in (\mathfrak{g}/\mathfrak{t}) \oplus \mathfrak{t}$. We choose our path by lineariation, such that

$$\mu(a) = exp(aX) \in G \text{ and } \nu(a) = exp(aS) \in T \text{ where } a \in [-1, 1]$$

So then, $\mu(0) = e$ and $\nu(0) = e$ with $\mu'(0) = X,\ \nu'(0) = S$. We define the map

$$f(a) = \Psi(\mu(a), \nu(a)) = \mu(a)\nu(a)\mu(a)^{-1} = exp(aX)exp(aS)exp(-aX)$$

Notice that this is a smooth map $f : [-1, 1] \to a$ with $f(0) = e$. We calculate by the product rule,

$$\frac{d}{da}f(a) = X \cdot exp(aX)exp(aS)exp(-aX) + exp(ax) \cdot S \cdot exp(aS)exp(-aX) - exp(aX)exp(aS)exp(-aX) \cdot X$$

at $a = 0$ gives us $f'(0) = X + S - X = S$. Hence, we have shown $d\Psi_{(eT,e)}(X, S) = S$. Now, for a simpler and directforward case with a relative decomposition of the Lie algebra $\mathfrak{p} \oplus \mathfrak{t}$; we can calculate the differential $d\Psi$ at (eT, e), given by

$$d\Psi_{(eT,e)}(X, S) = (Id - Ad_{\nu^{-1}})(X) + Ad_\nu(S) \text{ for } X \in \mathfrak{p}, S \in \mathfrak{t}$$

Since we already know that $T \subset G$ is abelian, and also $|detAd_\nu| = 1$, where the map $\Phi : G \to \mathbb{R}^+$ defined by $\mu \mapsto |detAd_\mu|$ is Lie group homomorphism. i.e For any adjoint action is the linear map $Ad_\mu : \mu \to G$, such that, $X \mapsto \mu X\mu^{-1}$, taking the determinant gives a function: $\mu \to |det(Ad_\mu)| = 1$. This is because our

Lie group G is compact, then a continous group homomorphism from a compact group to $\mathbb{R}^+$ must have its image to be a compact supgroup of $\mathbb{R}^+$, which is $\{1\}$. It follows to see

$$|det(d\Phi)_{(\mu T,\nu)}| = |det(Ad_{\nu^{-1}} - Id)|_{\mathfrak{p}} detAd_\nu| = |det(Ad_{\nu^{-1}} - Id)|_{\mathfrak{p}}|$$

so that we find

$$det(d\Phi)_{(\mu T,\nu)} = det(Ad_{\nu^{-1}} - Id)|_{\mathfrak{p}} \cdot detAd_\nu$$

Now, we investigate the regualarity and we deduce our focus into maximal torus $T \subset G$, where $|detAd_\nu| = 1$ for all $\nu \in T$. Geometrically, an element $\nu \in T$ is called *regular* if its centralizer in G is exactly T, and in a compact Lie group, the *regular* elements form a dense open subset. In fact, such determinant vanishes for irregular ones, which has a measure zero; via conjugation, most elements in T and G are *regular*. Then we obeserve that there exists open dense subsets $T^{reg} \subset T$ and $G^{reg} \subset G$. So that $det([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}) \neq 0$ on T^{reg}, and Φ is locally diffeomorphic. Furthermore, going back to our initial map Φ where we have

$$\Phi(\mu_1 T, \nu_1) = \Phi(\mu_2 T, \nu_2) \text{ if and only if } \nu_1, \nu_2 \in T \text{ be conjugates in } G$$

Equivalently, such conjugates lie in the same orbit of W, so Φ is $|W|-to-one$ covering map from $G/T \times T^{reg}$ to G^{reg}. i.e. for a fixed regular point $\nu \in T^{reg}$, its conjugacy class of G is $\{\mu\nu\mu^{-1} : \mu \in G\}$. Since ν is regular, every conjugate of ν arises exactly $|W|$ times from the different quotient G/T representative. As before, we have the map $\Phi : (G/T) \times T \to G$, defined by $(\mu T, \nu) \mapsto (\mu\nu\mu^{-1})$ where $\mu_1 T \mu_1^{-1} = \mu_2\nu_2\mu_2^{-1}$. So, wehn ν_1, ν_2 are regular points at T as they belong to dense open subsets of T^{reg}, their centralizers are exactly T itself. If so $\mu_1\nu_1\mu_1^{-1} = \mu_2\nu_2\mu_2^{-1}$, then $\nu_1 = (\mu_1^{-1}\mu_2)\nu_2(\mu_1^{-1}\mu_2)^{-1}$. Since $\nu_1, \nu_2 \in T$ is maximal torus, the only way they can be conjugated is whenever the conjugating elements $\mu_1^{-1}\mu_2 \in N(T)$. Finally, by using all the observation above, we move forward to the last step. For any class function f where $f(\mu\nu\mu^{-1}) = f(\nu)$, we may write,

$$\int_G f(\mu)d\mu = \int_{G/T\times T} f(\Phi(\mu T, \nu))dg, \text{ where } dg \text{ is the Haar measure on } G$$

Applying Jacobians of the change of variables since Φ is not volume preserving; we rewrite the formula as

$$\int_G f(\mu)d\mu = \frac{1}{|W|}\int_{G/T\times T^{reg}} f(\nu)|det[Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}|d(\mu T)d\nu$$

Here, we already know that the density $d(\mu T)$ on G/T is normalized, so that such integration over G/T contributes to a factor of 1. Hence, we obtain

$$\int_G f(\mu)d\mu = \frac{1}{|W|}\int_{T^{reg}} f(\nu)|det[Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}|d\nu$$

, and thus, since the set of irregular elements has a measure zero, which allows us to extend to all maximal torus T, we have shown the integration formula

$$\int_G f(\mu)d\mu = \frac{1}{|W|}\int_{T} f(\nu)|det[Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}|d\nu$$

□

Corollary 3.4. *We can extend our theorem above to the continous function. Let $f \in C(G)$ be any continous function on G. Then we obtain the same integration formula.*

Proof. Let us define $\tilde{f}$ on maximal torus T by averaging over the conjugacy, $\tilde{f}(\nu) = \int_G f(\mu\nu\mu^{-1})d\mu$. Clearly, this is $W - invariant$ function on T, and we can identify easily as a class function on G, which is now $\int_G f(\mu)d\mu = \int_G \tilde{f}(\mu)d\mu$. Hence, by the invariance of the *Haar measure*, we get

$$\begin{aligned}\int_G f(\mu)d\mu &= \frac{1}{|W|}\int_T f(\nu)det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right)\left(\int_G f(\mu\nu\mu^{-1})d\mu\right)d\nu \\ &= \frac{1}{|W|}\int_T f(\nu)det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right)\tilde{f}(\nu)d\nu\end{aligned}$$

□

Example 3.5. Let $G = U(n)$ with the maximal torus $T = \{diag(e^{i\nu_1, \cdots, e^{i\nu_n}})|\nu_j \in (0, 2\pi)\}$. Followed by the result of the theorem(3.3), $d\nu$ be the normalized *Haar measure* on T, and for each $\mu \in G$ is the conjugate to the diagonal matrix $\nu \in T$. So, we see that there exists $u \in U(n)$, such that, $\mu = u\nu u^{-1}$. We define the parametrization map

$$\Phi : U(n)/T \times T \to U(n) \text{ defined by } (uT, \nu) \mapsto (u\nu u^{-1})$$

Then, in order to ensure a volume preserving, we can reparamatrize up to Jacobian on the average $\frac{1}{|W|}$

$$\int_{U(n)} f(\mu)d\mu = \frac{1}{|W|}\int_T f(\nu)|det[Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}} d\nu$$

Notice that $U(n)$ consists of skew-Hermitian matrices and $\mathfrak{p}$ is the subspace of off-diagonal matrices. From this set up and know that the product of eigen values over all off-diagonal matrices with indices $\{j, k\}$, $\{k, j\}$ are unordered, we compute the determinant as $det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right) = \prod_{j<k} |exp(i\nu_j) - exp(i\nu_k)|^2$

$$\begin{aligned} det\left([Ad_{\nu^{-1}} - Id]|_{\mathfrak{p}}\right) &= \prod_{j \neq k} (exp(-i\nu_j)exp(i\nu_k) - 1) = \prod_{j<k} (exp(i\nu_j)exp(-i\nu_k) - 1)\left(exp(i\nu_k)exp(-i\nu_j) - 1\right) \\ &= \prod_{j<k} (exp(i\nu_j) - exp(i\nu_k))\left(exp(-i\nu_j) - exp(-i\nu_k)\right) \\ &= \prod_{j<k} |exp(i\nu_j) - exp(i\nu_k)|^2 \end{aligned}$$

But, we already know that $U(n) \cong S_n$, which implies $|W| = |n!|$. Thus, we have shown

$$\int_{U(n)} f(\mu)d\mu = \frac{1}{n!}\int_T f(\nu)\prod_{j<k} |exp(i\nu_j) - exp(i\nu_k)|^2 d\nu$$

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